Problem 19

In each of Problems 10 through 19, either compute the inverse of the given matrix, or else show that it is singular.

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{pmatrix}$$

Solution

Start by calculating the determinant.

$$\det \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{pmatrix} = 1 \begin{vmatrix} 2 & -4 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & -4 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 & 2 \\ 1 & 0 & 3 \\ -2 & 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 2 & -4 \\ 1 & 0 & 1 \\ -2 & 2 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} -4 & 2 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} -4 & 2 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} -1 & -4 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} - 6 \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= 2(-1) + 2(-14) - 2(-14) - 1(3) - 2(-6) - 6(2)$$

$$= -5$$

Since it's not zero, an inverse for the given matrix exists.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The aim is to make the left side 1's and 0's as the right side is now. Multiply the first row by 2 and add it to the second row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Multiply the second row by -1 and add it to the third row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Multiply the first row by 2 and add it to the fourth row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Add the fourth row to the third row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Divide the third row by 5.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply the third row by -4 and add it to the fourth row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & -1 & 2 & \frac{4}{5} & -\frac{4}{5} & \frac{1}{5} \end{array}\right)$$

Multiply the fourth row by -1.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Multiply the fourth row by -2 and add it to the second row.

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Switch the first row and the second row.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Multiply the first row by -1 and add it to the second row.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & -1 & 2 & 0 & -5 & -\frac{13}{5} & \frac{8}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Multiply the third row by -2 and add it to the second row.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & -1 & 0 & 0 & -5 & -\frac{11}{5} & \frac{6}{5} & -\frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Multiply the second row by -1.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

Therefore, the inverse of the given matrix is

$$\begin{pmatrix} 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$